

# On the Sum of Divisors of Mixed Powers

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**Abstract:** Let  $d(n)$  denote the Dirichlet divisor function. Define

$$\mathcal{S}_k(x) = \sum_{\substack{1 \leq n_1, n_2, n_3 \leq x^{1/2} \\ 1 \leq n_4 \leq x^{1/k}}} d(n_1^2 + n_2^2 + n_3^2 + n_4^k), \quad 3 \leq k \in \mathbb{N}.$$

In this paper, we establish an asymptotic formula of  $\mathcal{S}_k(x)$  and prove that

$$\mathcal{S}_k(x) = C_1(k)x^{3/2+1/k} \log x + C_2(k)x^{3/2+1/k} + O(x^{3/2+1/k-\delta_k+\varepsilon}),$$

where  $C_1(k)$ ,  $C_2(k)$  are two constants depending only on  $k$ , with  $\delta_3 = \frac{19}{60}$ ,  $\delta_4 = \frac{5}{24}$ ,  $\delta_5 = \frac{19}{140}$ ,  $\delta_6 = \frac{25}{192}$ ,  $\delta_7 = \frac{457}{4032}$ ,  $\delta_k = \frac{1}{k+2} + \frac{1}{2k^2(k-1)}$  for  $k \geq 8$ .

**Keywords:** Divisor function; circle method; mixed power; asymptotic formula

**Mathematics Subject Classification 2010:** 11P05, 11P32, 11P55

## 1 Introduction and main result

Let  $d(n)$  be the Dirichlet divisor function. Gafurov [2, 3] studied the number of divisors of a binary quadratic form and obtained the asymptotic formula

$$\sum_{1 \leq m, n \leq x} d(m^2 + n^2) = A_1 x^2 \log x + A_2 x^2 + O(x^{5/3} \log^9 x),$$

where  $A_1$  and  $A_2$  are certain constants. Later this result was improved by Yu [13], who gave the asymptotic formula

$$\sum_{1 \leq m, n \leq x} d(m^2 + n^2) = A_1 x^2 \log x + A_2 x^2 + O(x^{3/2+\varepsilon}),$$

for any fixed  $\varepsilon > 0$ .

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In 2000, Calderán and de Velasco [1] studied the divisors of the quadratic form  $m_1^2 + m_2^2 + m_3^2$  and established the asymptotic formula

$$\sum_{1 \leq m_1, m_2, m_3 \leq x} d(m_1^2 + m_2^2 + m_3^2) = \frac{8\zeta(3)}{5\zeta(5)} x^3 \log x + O(x^3). \quad (1.1)$$

In 2012, Guo and Zhai [5] improved (1.1) to

$$\sum_{1 \leq m_1, m_2, m_3 \leq x} d(m_1^2 + m_2^2 + m_3^2) = 2C_1 I_1 x^3 \log x + (C_1 I_2 + C_2 I_1) x^3 + O(x^{8/3+\varepsilon}),$$

where

$$\begin{aligned} C_1 &= \sum_{q=1}^{\infty} \frac{1}{q^4} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{h=1}^q e\left(\frac{ah^2}{q}\right) \right)^3, \\ C_2 &= \sum_{q=1}^{\infty} \frac{-2 \log q + 2\gamma}{q^4} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{h=1}^q e\left(\frac{ah^2}{q}\right) \right)^3, \\ I_1 &= \int_{-\infty}^{+\infty} \left( \int_0^1 e(\mu^2 \beta) d\mu \right)^3 \left( \int_0^3 e(-\mu \beta) d\mu \right) d\beta \end{aligned}$$

and

$$I_2 = \int_{-\infty}^{+\infty} \left( \int_0^1 e(\mu^2 \beta) d\mu \right)^3 \left( \int_0^3 e(-\mu \beta) \log \mu d\mu \right) d\beta.$$

Later, Zhao [14] improved the error term  $O(x^{8/3+\varepsilon})$  to  $O(x^2 \log^7 x)$ . Moreover, Lü and Mu [8] consider the nonhomogeneous case. They proved that, for  $k \geq 3$ , there holds

$$\sum_{\substack{1 \leq n_1, n_2 \leq x^{1/2} \\ 1 \leq n_3 \leq x^{1/k}}} d(n_1^2 + n_2^2 + n_3^k) = A(k) x^{1+1/k} \log x + B(k) x^{1+1/k} + O(x^{1+1/k-\theta(k)+\varepsilon}),$$

where  $A(k)$ ,  $B(k)$  are two constants depending only on  $k$ ,  $\theta(3) = 5/42$ ,  $\theta(4) = 1/16$ ,  $\theta(5) = 1/40$ ,  $\theta(k) = 1/(k2^{k-1})$  for  $6 \leq k \leq 7$  and  $\theta(k) = 1/(2k^2(k-1))$  for  $k \geq 8$ .

In 2014, Hu [6] considered the divisors of the quaternary form  $m_1^2 + m_2^2 + m_3^2 + m_4^2$  and obtained

$$\begin{aligned} \sum_{1 \leq m_1, m_2, m_3, m_4 \leq x} d(m_1^2 + m_2^2 + m_3^2 + m_4^2) \\ = 2\mathcal{C}_1 \mathcal{I}_1 x^4 \log x + (\mathcal{C}_1 \mathcal{I}_2 + \mathcal{C}_2 \mathcal{I}_1) x^4 + O(x^{7/2+\varepsilon}), \end{aligned} \quad (1.2)$$

where

$$\mathcal{C}_1 = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \right)^4,$$

$$\mathcal{C}_2 = \sum_{q=1}^{\infty} \frac{-2\log q + 2\gamma}{q^5} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \right)^4,$$

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \left( \int_0^1 e(u^2\lambda) du \right)^4 \left( \int_0^4 e(-u\lambda) du \right) d\lambda$$

and

$$\mathcal{I}_2 = \int_{-\infty}^{+\infty} \left( \int_0^1 e(u^2\lambda) du \right)^4 \left( \int_0^4 e(-u\lambda) \log u du \right) d\lambda.$$

Later, Liu and Hu [7] improved the error term  $O(x^{7/2+\varepsilon})$  to  $O(x^3 \log x)$ .

In this paper, we consider the nonhomogeneous case of the form  $n_1^2 + n_2^2 + n_3^2 + n_4^k$  and establish the following theorem.

**Theorem 1.1** *Let*

$$S_k(x) = \sum_{\substack{1 \leq n_1, n_2, n_3 \leq x^{1/2} \\ 1 \leq n_4 \leq x^{1/k}}} d(n_1^2 + n_2^2 + n_3^2 + n_4^k), \quad 3 \leq k \in \mathbb{N}.$$

*Then we have*

$$\mathcal{S}_k(x) = \mathfrak{S}_1 \mathfrak{J}_1 x^{3/2+1/k} \log x + (\mathfrak{S}_1 \mathfrak{J}_2 + \mathfrak{S}_2 \mathfrak{J}_1) x^{3/2+1/k} + O(x^{3/2+1/k-\delta_k+\varepsilon}),$$

*where*

$$\begin{aligned} \delta_3 &= \frac{19}{60}, & \delta_4 &= \frac{5}{24}, & \delta_5 &= \frac{19}{140}, & \delta_6 &= \frac{25}{192}, \\ \delta_7 &= \frac{457}{4032}, & \delta_k &= \frac{1}{k+2} + \frac{1}{2k^2(k-1)} \quad \text{for } k \geq 8, \end{aligned}$$

$$\mathfrak{S}_1 = \sum_{q=1}^{\infty} \frac{1}{q^5} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \right)^3 \left( \sum_{r=1}^q e\left(\frac{ar^k}{q}\right) \right),$$

$$\mathfrak{S}_2 = \sum_{q=1}^{\infty} \frac{-2\log q + 2\gamma}{q^5} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{r=1}^q e\left(\frac{ar^2}{q}\right) \right)^3 \left( \sum_{r=1}^q e\left(\frac{ar^k}{q}\right) \right),$$

$$\mathfrak{J}_1 = \int_{-\infty}^{+\infty} \left( \int_0^1 e(\alpha\mu^2) d\mu \right)^3 \left( \int_0^1 e(\alpha\mu^k) d\mu \right) \left( \int_0^3 e(-\alpha\mu) d\mu \right) d\alpha,$$

$$\mathfrak{J}_2 = \int_{-\infty}^{+\infty} \left( \int_0^1 e(\alpha\mu^2) d\mu \right)^3 \left( \int_0^1 e(\alpha\mu^k) d\mu \right) \left( \int_0^3 e(-\alpha\mu) \log \mu d\mu \right) d\alpha.$$

**Notation.** Throughout this paper,  $x$  always denotes a sufficiently large real number;  $\varepsilon$  always denotes an arbitrary small positive constant, which may not be the same

at different occurrences.  $e(x) = e^{2\pi ix}$ ;  $f(x) \ll g(x)$  means that  $f = O(g(x))$ . For the sake of brevity, we define

$$\mathfrak{J}_1(\beta) = \left( \int_0^1 e(\beta\mu^2) d\mu \right)^3 \left( \int_0^1 e(\beta\mu^k) d\mu \right) \left( \int_0^3 e(-\beta\mu) d\mu \right)$$

and

$$\mathfrak{J}_2(\beta) = \left( \int_0^1 e(\beta\mu^2) d\mu \right)^3 \left( \int_0^1 e(\beta\mu^k) d\mu \right) \left( \int_0^3 e(-\beta\mu) \log \mu d\mu \right).$$

## 2 Preliminary Lemmas

For any  $\alpha \in \mathbb{R}$ , define

$$f_\ell(\alpha) = \sum_{1 \leq n \leq x^{1/\ell}} e(\alpha n^\ell), \quad f(\alpha) = \sum_{1 \leq n \leq 4x} d(n) e(\alpha n).$$

**Lemma 2.1** *For any real numbers  $\alpha$  and  $\tau \geq 1$ , there exist integers  $a$  and  $q$ ,  $(a, q) = 1$ ,  $1 \leq q \leq \tau$ , such that*

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leq \frac{1}{q\tau}.$$

**Proof.** See C. D. Pan and C. B. Pan [9], Lemma 5.19. ■

Let

$$\log x < 2Q < \tau < x, \quad Q\tau \asymp x, \quad Q \ll x^{2/(k+2)}. \quad (2.1)$$

For any  $1 \leq a < q \leq Q$  with  $(a, q) = 1$ , define

$$\mathfrak{M}(a, q) = \left[ \frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right]$$

and

$$\mathfrak{M} = \bigcup_{q \leq Q} \bigcup_{\substack{1 \leq a \leq q \\ (a, q) = 1}} \mathfrak{M}(a, q), \quad \mathfrak{m} = \left[ \frac{1}{\tau}, 1 + \frac{1}{\tau} \right] \setminus \mathfrak{M}.$$

We call  $\mathfrak{M}$  the major arc and  $\mathfrak{m}$  the minor arc. By the definition of  $\mathcal{S}_k(x)$  and orthogonality of exponential function, we have

$$\begin{aligned} \mathcal{S}_k(x) &= \int_0^1 f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\ &= \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\ &=: \mathcal{S}_k(x, \mathfrak{M}) + \mathcal{S}_k(x, \mathfrak{m}). \end{aligned}$$

**Lemma 2.2** *For any  $a, q \in \mathbb{Z}$  with  $(a, q) = 1$  and  $q > 0$ , let*

$$S_k(q, a) = \sum_{r=1}^q e\left(\frac{ar^k}{q}\right).$$

Then we have

$$S_k(q, a) \ll q^{(k-1)/k}.$$

**Proof.** See Vaughan [11], Theorem 4.2. ■

**Lemma 2.3** For  $k \geq 1$ , we have

$$\int_0^1 e(\beta \mu^k) d\mu \ll \frac{1}{(1 + |\beta|)^{1/k}}, \quad (2.2)$$

$$\int_0^3 e(\beta \mu^k) \log \mu d\mu \ll \frac{\log(2 + |\beta|)}{1 + |\beta|}. \quad (2.3)$$

**Proof.** See Lü and Mu [8], Lemma 1.2. ■

**Lemma 2.4** Suppose that  $(a, q) = 1$  and  $\alpha = \frac{a}{q} + \beta$ . Then

$$f_k(\alpha) = V_k(\alpha, q, a) + O(q^{1/2+\varepsilon}(1 + x|\beta|)^{1/2}).$$

Moreover, if  $|\beta| \leq \frac{x^{1/k-1}}{2kq}$ , then

$$f_k(\alpha) = V_k(\alpha, q, a) + O(q^{1/2+\varepsilon}),$$

where

$$V_k(\alpha, q, a) = x^{1/k} \frac{S_k(q, a)}{q} \int_0^1 e(x\beta \mu^k) d\mu.$$

**Proof.** See Vaughan [11], Theorem 4.1. ■

**Lemma 2.5** Suppose that  $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}$  and  $Q\tau \leq x$ ,  $\tau > x^{1/2+\varepsilon}$ . Then

$$\begin{aligned} f(-\alpha) &= \frac{x \log x}{q} \int_0^3 e(-\mu x \beta) d\mu + \frac{x}{q} \int_0^3 e(-\mu x \beta) \log \mu d\mu \\ &\quad + \frac{-2 \log q + 2\gamma}{q} x \int_0^3 e(-\mu x \beta) d\mu + O(\Delta), \end{aligned}$$

where

$$\Delta = x^\varepsilon (q^{1/2} x \tau^{-1} + q^{2/3} x^{1/3}). \quad (2.4)$$

**Proof.** See Guo and Zhai [5], Lemma 7.1. ■

**Lemma 2.6** Suppose that

$$L(H) = \sum_{i=1}^m A_i H^{a_i} + \sum_{j=1}^n B_j H^{b_j},$$

where  $A_i$ ,  $B_j$ ,  $a_i$  and  $b_j$  are positive. Assume that  $H_1 \leq H_2$ . Then there exists some  $\mathcal{H}$  with  $H_1 \leq \mathcal{H} \leq H_2$  and

$$L(\mathcal{H}) \ll \sum_{i=1}^m A_i H_1^{a_i} + \sum_{j=1}^n B_j H_2^{b_j} + \sum_{i=1}^m \sum_{j=1}^n (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)}.$$

The implied constant depends only on  $m$  and  $n$ .

**Proof.** See Srinivasan [10], Lemma 3 or Graham and Kolesnik [4], Lemma 2.4. ■

**Lemma 2.7** *Suppose that*

$$(a, q) = 1, \quad \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}, \quad \phi(x) = \alpha x^k + \alpha_1 x^{k-1} + \cdots + \alpha_{k-1} x + \alpha_k.$$

*Then*

$$\sum_{1 \leq x \leq Y} e(\phi(x)) \ll Y^{1+\varepsilon} (q^{-1} + Y^{-1} + qY^{-k})^{1/2^{k-1}}.$$

**Proof.** See Vaughan [11], Lemma 2.4. ■

**Lemma 2.8** *Suppose that  $1 \leq j \leq k$ . Then*

$$\int_0^1 \left| \sum_{m=1}^Y e(\alpha m^k) \right|^{2j} d\alpha \ll Y^{2j-j+\varepsilon}.$$

**Proof.** See Vaughan [11], Lemma 2.5. ■

**Lemma 2.9** *Let  $j$  be an integer with  $j \geq 2$ . Suppose that there exist integers  $a, q$  with  $q \geq 1$ ,  $(a, q) = 1$  such that  $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}$  and  $q \leq x$ . Then one has*

$$f_j(\alpha) \ll x^{1/j+\varepsilon} (q^{-1} + x^{-1/j} + qx^{-1})^{1/2j(j-1)}.$$

**Proof.** See Wooley [12], Theorem 1.5. ■

### 3 Proof of Theorem 1.1

It is obvious that

$$\mathcal{S}_k(x, \mathfrak{M}) = \sum_{1 \leq q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \int_{\frac{a}{q} - \frac{1}{q^\tau}}^{\frac{a}{q} + \frac{1}{q^\tau}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha.$$

For  $\alpha = \frac{a}{q} + \beta \in \mathfrak{M}$ , by Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned} & f_2^3(\alpha) f_k(\alpha) f(-\alpha) \\ &= \left( V_2(\alpha, q, a) + O(q^{1/2+\varepsilon}) \right)^3 \left( V_k(\alpha, q, a) + O(q^{1/2+\varepsilon} (1 + x|\beta|)^{1/2}) \right) \\ & \times \left( \frac{x \log x}{q} \int_0^3 e(-\mu x \beta) d\mu + \frac{x}{q} \int_0^3 e(-\mu x \beta) \log \mu d\mu \right. \\ & \quad \left. + \frac{-2 \log q + 2\gamma}{q} x \int_0^3 e(-\mu x \beta) d\mu + O(\Delta) \right) \end{aligned}$$

$$\begin{aligned}
&= x^{3/2+1/k} \frac{S_2^3(q, a) S_k(q, a)}{q^4} \left( \int_0^1 e(x\beta\mu^2) d\mu \right)^3 \left( \int_0^1 e(x\beta\mu^k) d\mu \right) \\
&\quad \times \left( \frac{x \log x}{q} \int_0^3 e(-\mu x\beta) d\mu + \frac{x}{q} \int_0^3 e(-\mu x\beta) \log \mu d\mu \right. \\
&\quad \left. + \frac{-2 \log q + 2\gamma}{q} x \int_0^3 e(-\mu x\beta) d\mu \right) + O(x^{5/2+\varepsilon} q^{-2} (1+x|\beta|)^{-2}) \\
&\quad + O(x^{1+1/k+\varepsilon} q^{1/2-1/k} (1+x|\beta|)^{-1-1/k} + x^{1+\varepsilon} q (1+x|\beta|)^{-1/2}) \\
&\quad + O\left(\Delta(x^{3/2+1/k} q^{-3/2-1/k} (1+x|\beta|)^{-3/2-1/k} + q^2 x^\varepsilon (1+x|\beta|)^{1/2} \right. \\
&\quad \left. + x^{1/k+\varepsilon} q^{3/2-1/k} (1+x|\beta|)^{-1/k} + x^{3/2+\varepsilon} q^{-1} (1+x|\beta|)^{-1}\right),
\end{aligned}$$

where  $\Delta$  is defined in (2.4). Then we have

$$\begin{aligned}
&\int_{\frac{a}{q} - \frac{1}{q\tau}}^{\frac{a}{q} + \frac{1}{q\tau}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\
&= \int_{-\frac{1}{q\tau}}^{\frac{1}{q\tau}} f_2^3\left(\frac{a}{q} + \beta\right) f_k\left(\frac{a}{q} + \beta\right) f\left(-\frac{a}{q} - \beta\right) d\beta \\
&= \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{5/2+1/k} \log x \\
&\quad \times \int_{-\frac{1}{q\tau}}^{\frac{1}{q\tau}} \left( \int_0^1 e(x\beta\mu^2) d\mu \right)^3 \left( \int_0^1 e(x\beta\mu^k) d\mu \right) \left( \int_0^3 e(x\beta\mu) d\mu \right) d\beta \\
&\quad + \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{5/2+1/k} \\
&\quad \times \int_{-\frac{1}{q\tau}}^{\frac{1}{q\tau}} \left( \int_0^1 e(x\beta\mu^2) d\mu \right)^3 \left( \int_0^1 e(x\beta\mu^k) d\mu \right) \left( \int_0^3 e(x\beta\mu) \log \mu d\mu \right) d\beta \\
&\quad + (-2 \log q + 2\gamma) \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{5/2+1/k} \\
&\quad \times \int_{-\frac{1}{q\tau}}^{\frac{1}{q\tau}} \left( \int_0^1 e(x\beta\mu^2) d\mu \right)^3 \left( \int_0^1 e(x\beta\mu^k) d\mu \right) \left( \int_0^3 e(x\beta\mu) d\mu \right) d\beta \\
&\quad + O(x^{3/2+\varepsilon} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k+\varepsilon} q^{-1-1/k} \tau^{-1} \\
&\quad + x^{5/6+1/k+\varepsilon} q^{-5/6-1/k} + x^{3/2+\varepsilon} q^{-1/2} \tau^{-1} + x^{5/6+\varepsilon} q^{-1/3}) \\
&= \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \log x \int_{-\frac{x}{q\tau}}^{\frac{x}{q\tau}} \mathfrak{J}_1(\beta) d\beta \\
&\quad + \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \int_{-\frac{x}{q\tau}}^{\frac{x}{q\tau}} \mathfrak{J}_2(\beta) d\beta \\
&\quad + (-2 \log q + 2\gamma) \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \int_{-\frac{x}{q\tau}}^{\frac{x}{q\tau}} \mathfrak{J}_1(\beta) d\beta \\
&\quad + O(x^{3/2+\varepsilon} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k+\varepsilon} q^{-1-1/k} \tau^{-1} \\
&\quad + x^{5/6+1/k+\varepsilon} q^{-5/6-1/k} + x^{3/2+\varepsilon} q^{-1/2} \tau^{-1} + x^{5/6+\varepsilon} q^{-1/3}). \tag{3.1}
\end{aligned}$$

Applying Lemma 2.3, we have

$$\mathfrak{J}_1(\beta) \ll \frac{1}{(1 + |\beta|)^{5/2+1/k}}, \quad \mathfrak{J}_2(\beta) \ll \frac{\log(2 + |\beta|)}{(1 + |\beta|)^{5/2+1/k}}.$$

Therefore, we have

$$\int_{|\beta| > \frac{x}{q\tau}} \mathfrak{J}_1(\beta) d\beta \ll \int_{|\beta| > \frac{x}{q\tau}} \frac{1}{(1 + |\beta|)^{5/2+1/k}} d\beta \ll \frac{1}{\left(1 + \frac{x}{q\tau}\right)^{3/2+1/k}} \quad (3.2)$$

and

$$\int_{|\beta| > \frac{x}{q\tau}} \mathfrak{J}_2(\beta) d\beta \ll \int_{|\beta| > \frac{x}{q\tau}} \frac{\log(2 + |\beta|)}{(1 + |\beta|)^{5/2+1/k}} d\beta \ll \frac{\log\left(2 + \frac{x}{q\tau}\right)}{\left(1 + \frac{x}{q\tau}\right)^{3/2+1/k}}. \quad (3.3)$$

Hence, from (3.1), (3.2) and (3.3), we get

$$\begin{aligned} & \int_{\frac{a}{q} - \frac{1}{q\tau}}^{\frac{a}{q} + \frac{1}{q\tau}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\ = & \mathfrak{J}_1 \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \log x + \mathfrak{J}_2 \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \\ & + (-2 \log q + 2\gamma) \mathfrak{J}_1 \frac{S_2^3(q, a) S_k(q, a)}{q^5} x^{3/2+1/k} \\ & + O\left(x^\varepsilon q^{-1} \tau^{3/2+1/k} + x^{3/2+\varepsilon} q^{-2} + x^{1/k+\varepsilon} q^{1/2-1/k} + x^{3/2+1/k+\varepsilon} q^{-1-1/k} \tau^{-1} \right. \\ & \left. + x^{5/6+1/k+\varepsilon} q^{-5/6-1/k} + x^{3/2+\varepsilon} q^{-1/2} \tau^{-1} + x^{5/6+\varepsilon} q^{-1/3}\right). \end{aligned} \quad (3.4)$$

By Lemma 2.2, it follows that

$$\sum_{q > Q} \sum_{\substack{a=1 \\ (a, q)=1}}^q \frac{S_2^3(q, a) S_k(q, a)}{q^5} \ll \sum_{q > Q} q^{-3/2-1/k} \ll Q^{-1/2-1/k}. \quad (3.5)$$

Therefore, from (3.4) and (3.5), we obtain

$$\begin{aligned} \mathcal{S}_k(x, \mathfrak{M}) &= \sum_{1 \leq q \leq Q} \sum_{\substack{a=1 \\ (a, q)=1}}^q \int_{\frac{a}{q} - \frac{1}{q\tau}}^{\frac{a}{q} + \frac{1}{q\tau}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\ &= \mathfrak{S}_1 \mathfrak{J}_1 x^{3/2+1/k} \log x + (\mathfrak{S}_1 \mathfrak{J}_2 + \mathfrak{S}_2 \mathfrak{J}_1) x^{3/2+1/k} \\ &\quad + O\left(x^{3/2+1/k+\varepsilon} Q^{-1/2-1/k} + x^\varepsilon Q \tau^{3/2+1/k} + x^{1/k+\varepsilon} Q^{5/2-1/k} \right. \\ &\quad \left. + x^{3/2+1/k+\varepsilon} Q^{1-1/k} \tau^{-1} + x^{5/6+1/k+\varepsilon} Q^{7/6-1/k} \right. \\ &\quad \left. + x^{3/2+\varepsilon} Q^{3/2} \tau^{-1} + x^{5/6+\varepsilon} Q^{5/3}\right). \end{aligned} \quad (3.6)$$

It remains to estimate the integral on the minor arc  $\mathfrak{m}$ . At this time, for  $\alpha \in \mathfrak{m}$ , we have  $Q < q \leq \tau$ . We consider four different cases as follows.



**Case 1.** If  $3 \leq k \leq 5$ , by noting the fact that  $Q\tau \asymp x$ ,  $Q < x^{2/(k+2)}$  and Lemma 2.7, we have

$$\begin{aligned} f_2(\alpha) &\ll x^{1/2+\varepsilon}(q^{-1} + x^{-1/2} + qx^{-1})^{1/2} \\ &\ll x^{1/2+\varepsilon}(Q^{-1/2} + x^{-1/4} + \tau^{1/2}x^{-1/2}) \\ &\ll x^{1/2+\varepsilon}Q^{-1/2}. \end{aligned} \quad (3.7)$$

Also, we have

$$\int_0^1 |f(-\alpha)|^2 d\alpha = \sum_{n \leq 4x} d^2(n) \ll x \log^3 x. \quad (3.8)$$

Therefore, it follows from Hölder's inequality, Lemma 2.8 and (3.8) that

$$\begin{aligned} \mathcal{S}_k(x, \mathfrak{m}) &= \int_{\mathfrak{m}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\ &\ll \max_{\alpha \in \mathfrak{m}} |f_2(\alpha)|^2 \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/4} \left( \int_0^1 |f_k(\alpha)|^4 d\alpha \right)^{1/4} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2} \\ &\ll x^{1+\varepsilon} Q^{-1} \cdot x^{3/4+1/(2k)+\varepsilon} \ll x^{7/4+1/(2k)+\varepsilon} Q^{-1}. \end{aligned} \quad (3.9)$$

**Case 2.** If  $k = 6$ , from Lemma 2.7 we have

$$\begin{aligned} f_6(\alpha) &\ll x^{1/6+\varepsilon}(Q^{-1/32} + x^{-1/192} + \tau^{1/32}x^{-1/32}) \\ &\ll x^{1/6+\varepsilon}(Q^{-1/32} + x^{-1/192}). \end{aligned} \quad (3.10)$$

Therefore, it follows from Cauchy's inequality, Lemma 2.8 and (3.8) that

$$\begin{aligned} \mathcal{S}_6(x, \mathfrak{m}) &= \int_{\mathfrak{m}} f_2^3(\alpha) f_6(\alpha) f(-\alpha) d\alpha \\ &\ll \max_{\alpha \in \mathfrak{m}} |f_2(\alpha)| \cdot \max_{\alpha \in \mathfrak{m}} |f_6(\alpha)| \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2} \\ &\ll x^{5/3+\varepsilon} Q^{-17/32} + x^{319/192+\varepsilon} Q^{-1/2}. \end{aligned} \quad (3.11)$$

**Case 3.** If  $k = 7$ , from Lemma 2.7 we have

$$\begin{aligned} f_7(\alpha) &\ll x^{1/7+\varepsilon}(Q^{-1/64} + x^{-1/448} + \tau^{1/64}x^{-1/64}) \\ &\ll x^{1/7+\varepsilon}(Q^{-1/64} + x^{-1/448}). \end{aligned} \quad (3.12)$$

Therefore, it follows from Cauchy's inequality, Lemma 2.8 and (3.8) that

$$\begin{aligned} \mathcal{S}_7(x, \mathfrak{m}) &= \int_{\mathfrak{m}} f_2^3(\alpha) f_7(\alpha) f(-\alpha) d\alpha \\ &\ll \max_{\alpha \in \mathfrak{m}} |f_2(\alpha)| \cdot \max_{\alpha \in \mathfrak{m}} |f_7(\alpha)| \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2} \\ &\ll x^{23/14+\varepsilon} Q^{-33/64} + x^{735/448+\varepsilon} Q^{-1/2}. \end{aligned} \quad (3.13)$$

**Case 4.** If  $k \geq 8$ , from Lemma 2.9 we have

$$\begin{aligned} f_k(\alpha) &\ll x^{1/k+\varepsilon} (Q^{-1} + x^{-1/k} + \tau x^{-1})^{1/(2k(k-1))} \\ &\ll x^{1/k+\varepsilon} (Q^{-1/(2k(k-1))} + x^{-1/(2k^2(k-1))}). \end{aligned} \quad (3.14)$$

Therefore, it follows from Hölder's inequality, Lemma 2.8 and (3.8) that

$$\begin{aligned} S_k(x, \mathfrak{m}) &= \int_{\mathfrak{m}} f_2^3(\alpha) f_k(\alpha) f(-\alpha) d\alpha \\ &\ll \max_{\alpha \in \mathfrak{m}} |f_2(\alpha)| \cdot \max_{\alpha \in \mathfrak{m}} |f_k(\alpha)| \left( \int_0^1 |f_2(\alpha)|^4 d\alpha \right)^{1/2} \left( \int_0^1 |f(-\alpha)|^2 d\alpha \right)^{1/2} \\ &\ll x^{3/2+1/k+\varepsilon} Q^{-(k^2-k+1)/(2k(k-1))} + x^{3/2+1/k-1/(2k^2(k-1))+\varepsilon} Q^{-1/2}. \end{aligned} \quad (3.15)$$

The rest thing that we need to do is to choose optimal parameters  $\tau$  and  $Q$ . By noting the condition (2.1), we can use  $xQ^{-1}$  to substitute  $\tau$  in (3.6). Then, by a simple calculation, it is easy to use Lemma 2.6 to obtain the desired asymptotic formula of  $S_k(x)$ . This completes the proof of Theorem 1.1.

### Acknowledgement

The authors would like to express the most and the greatest sincere gratitude to Professor Wenguang Zhai for his valuable advice and constant encouragement.

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